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A preference-free formula to value commodity derivatives*

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Abstract

This paper studies a new model of commodity prices in which the stochastic convenience yield is an affine function of past commodity returns. While preserving market completeness, the model exhibits price nonstationarity and mean reversion under the martingale measure, and, as a consequence, it is able to fit a slowly decaying term structure of futures return volatilities. The model nests mean reversion in levels and geometric Brownian motion, and renders preference-free formulas for the prices of futures contracts and European options.

Keywords: Commodity, Derivatives, Mean Reversion

JEL Classification: G13

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1 Introduction

This paper studies a new model of commodity prices in which the stochastic convenience yield is an affine function of a weighted average of past commodity returns. While preserving market completeness, the model exhibits price nonstationarity and mean reversion[1] under the martingale measure, and, as a consequence, it is able to fit a slowly decaying term structure of futures return volatilities. The model has mean reversion in levels[2] and geometric Brownian motion[3] as special cases, and therefore it can be seen as a generalization of these two commonly used one-factor models. For futures prices and the prices of European options, the model renders closed form solutions that do not depend on the price of risk of any of its state variables, and are therefore preference-free.

Commodity prices have certain empirical characteristics, such as seasonality, spikes, and mean reversion, that distinguish them from the prices of stocks and bonds. Seasonal patterns appear as a response of supply and demand to cyclical fluctuations due mainly to changes in weather. Spikes are the result of random shocks in markets in which the supply is relatively fixed in the short run. Mean reversion, on the other hand, arises as free entry and exit in competitive markets forces prices to gravitate towards the minimum average cost of production. As it reflects a phenomenon affecting commodities as a class, mean reversion is probably the most pervasive of all empirical characteristics of commodities. Most commodity models will certainly exhibit some form of mean reversion, even if they do not include seasonal patterns or jumps[4].

There exists a rich array of models aimed to capture the complex dynamics of commodity prices. Gibson and Schwartz (1990) introduced a model that combines nonstationarity and mean reversion through a stochastic convenience yield (see also Schwartz (1997)). Most of the literature that followed can be seen as an extension of Gibson and Schwartz (1990) seminal paper. To mention just a few representative examples, Hilliard and Reis (1998) add jumps to the spot price through a Poisson component[5]. Sorensen

[1] In this paper I use the expression "mean reversion" to refer to negative autocorrelation of returns generated by a temporary component in the spot price. This use of words is common in the literature (see Schwartz (1997)). I distinguish situations in which shocks partially vanish in the long run, from situations in which shocks totally vanish in the long run. In the first case I use the expression "mean reversion"; in the second case I use the expression "mean reversion in levels".

[2] This model is discussed in Bjerksund and Ekert (1995).

[3] In the commodities literature, geometric Brownian motion is associated to the works of Black (1976) and Brennan and Schwartz (1985), and refers to a model in which both the total return on the spot and the convenience yield are contant.

[4] See Schwartz (1997) for a colection of models with this characteristic.

[5] See also Cassassus and Dufresne (2005).

(2002) and Richter and Sorensen (2002) combine seasonal effects and stochastic volatility, and apply the model to the study of agricultural futures markets. Yan (2002) incoporates stochastic volatility and jumps in both the spot price and the spot volatility. In a recent study, using oil, copper, gold and silver data, Casassus and Dufresne (2005) find that three factors are needed to describe the dynamics of futures prices. However, and perhaps due to the very complexity that makes them so successful in capturing key features of data, multifactor models "have been adopted rather slowly by practitioners[6]".

One factor models, such as geometric Brownian motion (Black (1976), Brennan and Schwartz (1985)) and mean reversion in levels (Bjerksund and Ekert (1995), Schwartz (1997) model 1), may look too simple in comparison, but are still popular in the industry. Their popularity is partly explained by the tendency of practitioners to use models as means of extrapolating prices of liquid instruments to prices of illiquid instruments. This tendency creates a strong demand for parsimonious models. Market completeness is another reason that makes one-factor models popular. Under market completeness, unique option prices can be obtained by a simple arbitrage argument, and it is also possible to hedge a derivatives position using just the underlying asset and a bond. A complete market model may also prove useful in the management of a derivatives book. In addition, both Geometric Brownian motion and mean reversion in levels make it possible to obtain closed form solutions for futures prices and the prices of European options, although these solutions, in the case of mean reversion in levels, are not in general preference-free (see Schwartz (1997)).

One-factor models, on the other hand, have their own shortcomings. First, they imply that futures prices are perfectly correlated at all maturities, a prediction that is not supported by the data. Futures prices are in general imperfectly correlated, with correlations decreasing steadily with maturity. Second, one-factor models are unable to fit the term structure of futures return volatilities. For commodities, the term structure of futures return volatilities is negatively sloped, a stylized fact that can be explained by mean reversion (see Bessembinder, Coghenour, Seguin and Smoller (1996)). However, although volatilities go down uniformly as maturity increases, they do not seem to converge to zero, which suggests that random shocks to prices are only partially reversed in the long run. Geometric Brownian motion implies a flat term structure, while volatilities in the mean reversion in levels model converge to zero too quickly. One factor models may still be useful for derivatives that do not depend on the correlation between different futures prices, or when the maturities of the futures prices involved are not too far from each

[6] Cortazar and Schwartz (2003), page 216.

other. The inability to fit the term structure of futures return volatilities is more serious, because most derivatives on commodities use futures as the underlying asset, and so it is important for accurate valuation that models fit this term structure properly.

On this regard, one of the most important features of the model introduced in this paper is precisely its ability to fit a slowly decaying term structure of futures return volatilities. The reason for this is that in the model random shocks only partially vanish in the long run, which produces simultaneously nonstationarity and mean reversion. In the literature, these two features have only been obtained under market incompleteness, by means of a stochastic convenience yield imperfectly correlated to spot returns. Nonstationarity and mean reversion are obtained in this paper within a fully specified model that preserves market completeness.

The key assumption of the model is that the convenience yield is an affine function of a weighted sum of past commodity returns[7]. This assumption makes the convenience yield perfectly correlated with spot returns, and is the source of market completeness in the model. Assuming perfect correlation seems reasonable for certain commodities, like oil and copper, which exhibit strong comovement between convenience yield and spot returns. For example, using 10 years of weekly data, Schwartz (1997) finds a correlation between 70 and 90% for oil, depending on the subperiod analyzed, and 82% for copper. In a more recent study, Casassus and Dufresne (2005) find 79% and 78% for oil and copper, respectively. On the other hand, the convenience yield can be seen as a construction used to generate simultaneously mean reversion and nonstationarity under the risk-neutral measure, and the model introduced in this paper shows that it is possible to have both in a complete market.

The model also renders preference-free derivative prices. Preference-free prices are not just a consequence of market completeness. Lioui (2006) has studied the problem of pricing derivatives in complete markets in which the stock pays a stochastic dividend yield, an concluded that, even if there is a single source of uncertainty, a risk premium will appear in the derivatives formulas. This paper shows that this conclusion is not general, and presents a parameterization of the convenience yield (equivalent to the dividend yield in Lioui) that renders preference free derivatives formulas.

The model is estimated on a sample of oil futures prices. Oil is one of the most important traded commodities, and it has been widely studied in the literature. It has

[7] Cassasus and Dufresne (2005) have shown that dependence of the convenience yield on the spot price is needed to capture the spot price dynamics. The model studied in this paper uses a more general measure of performance, based on the history of past returns.

also been shown to exhibit mean reversion under the martingale measure (see Casassus and Dufresne (2005)). In assessing a commodity model, two metrics are of interest: i) how well the model captures a given term structure of future prices, ii) how well the model fits a given term structure of futures return volatilities. Information obtained from i) can be useful to value real options and other long term commitments. Results from ii) can be used in the valuation of American options, exotics, and options written on futures. Due to the known difficulties in the observation of spot prices, I estimate the model parameters using the Kalman filter (see Schwartz (1997)). The model produces reasonable pricing errors that are up to 50% lower than when mean reversion in levels is assumed. Next, I show that it is possible to parameterize the model to make it consistent with any given term structure of futures prices, and then I calibrate it to the implied term structure of futures volatilities. I find that the model generates a perfect fit; in contrast, assuming mean reversion in levels underestimates volatility at both ends of the term structure curve.

Practitioners usually estimate volatilities by calculating their implied values from the prices of a set of liquid options and the prices derived from the Black-Scholes formula. These implied volatilities are then used to price less liquid contracts. If the underlying asset exhibits mean reversion under the martingale measure, this procedure will overestimate volatilities -and prices- especially for longer term contracts. Imposing mean reversion in levels is a step towards the solution of this problem, but it may lead to the underestimation of volatilities when shocks to the underlying do not vanish completely in the long run. This seems to be the case for most commodities. This paper contributes to the literature by introducing a complete market model that is nonstationary and mean reverting under the martingale measure, and that is capable to fit the term structure of futures return volatilities. The model renders preference free formulas for futures and European option prices, and therefore it provides a useful benchmark to value more complex contracts for which no closed form solutions are known.

The structure of the paper is as follows. The model is presented in section 2. The price distribution under the martingale measure is obtained in section 3. Futures and option prices are derived in section 4, together with the riskless hedge and the most important "Greeks" (see also the Appendix). Section 5 presents empirical results. Finally, section 6 concludes.

2 Commodity price dynamics

Let's assume a frictionless financial market in which trading is continuous . The commodity spot price S_t satisfies the following differential equation:

$$\frac{dS_t}{S_t} = (\mu - \delta_t)\, dt + \sigma dW_t, \tag{1}$$

where μ is the total instantaneous expected return on the spot, δ_t is the stochastic convenience yield[8], and σ is the instantaneous return volatility. The only source of risk in the economy is a standard Wiener process, W_t, defined on a filtered probability space $(\Omega, \mathcal{F}, \Pi)$.

In this model, as it is common in the literature since the seminal Gibson and Schwartz (1991) paper, mean reversion in the spot is induced by a stationary convenience yield that is positively correlated to spot returns. The convenience yield reflects market views about the future scarcity of the commodity. When the market is tight, with strong demand and raising prices, the convenience yield is high; when the market is loose, with weak demand and falling prices, the convenience yield is low. Consistent with this, empirical work has found that, at least for certain commodities, the returns on the spot and the convenience yield are highly correlated. For example, Schwartz (1997) and, more recently, Casassus and Dufresne (2005) find that in the case of oil this correlation is about 80%.

Let $s_t = \log(S_t)$. Then, from equation (1):

$$ds_t = \left(\mu - \frac{1}{2}\sigma^2 - \delta_t\right) dt + \sigma dW_t, \tag{2}$$

Now define m_t as a weighted sum of past spot log returns, as:

$$m_t = \int_0^t e^{-\omega(t-u)} ds_u, \tag{3}$$

where $\omega \geq 0$ determines the weight of past returns. Next, to capture the dependence of the convenience yield on the price performance of the spot price, let's assume that the convenience yield is an affine function of m_t in the following way:

$$\delta_t = \delta + \phi m_t, \tag{4}$$

[8] The convenience yield is defined (Brennan (1958)) as the benefit, net of storage costs, that accrues to the holder of inventories rather than to the owner of a derivative contract written on the commodity.

where δ is a constant[9] and $\phi \geq 0$ is the loading of performance on the convenience yield. That is, in the case in which the inequality is strict, the convenience yield is positively related to performance, as the theory of storage suggests. On the other hand, $\phi = 0$ gives a constant convenience yield.

After these definitions, equation (2) can be rewritten as:

$$ds_t = \left(\mu - \frac{1}{2}\sigma^2 - \delta\right) dt - \phi m_t dt + \sigma dW_t, \tag{5}$$

Equations (3) and (5) determine endogenously the dynamics of m_t. Differentiating both sides of equation (3) gives:

$$dm_t = ds_t - \omega m_t dt, \tag{6}$$

which, after inserting equation (5), becomes:

$$dm_t = -\left(\omega + \phi\right)\left(m_t - \theta\right) dt + \sigma dW_t, \tag{7}$$

where $\theta = \frac{\mu - \frac{1}{2}\sigma^2 - \delta}{\omega + \phi}$. That is, m_t follows an Ornstein-Uhlenbeck process with long run mean θ and mean reversion speed $\omega + \phi$. Equations (5) and (7) describe the evolution of the spot price and the convenience yield. Note that, although (7) depends on the history of the spot, the system (S_t, m_t) is Markovian.

Two polar cases, $\phi = 0$ and $\omega = 0$, are of interest. When $\phi = 0$, the convenience yield is constant, and so the spot price follows geometric Brownian motion. This parameterization gives Brennan and Schwartz (1985) model[10]. When $\phi > 0$ and $\omega = 0$, $m_t = \log(S_t) - \log(S_0)$, and the process described in equation (5) is mean reverting in levels, with reversion rate ϕ and long-run mean:

$$\begin{aligned}\widetilde{\theta} &= \frac{\mu - \frac{1}{2}\sigma^2 - \delta + \phi s_0}{\phi} \\ &= \frac{\mu - \frac{1}{2}\sigma^2 - \widetilde{\delta}}{\phi}\end{aligned} \tag{8}$$

[9] This assumption is relaxed in section 4.2, where it is shown that by making δ a deterministic function of time, the model can be made consistent with any given term structure of futures prices.

[10] See also Black (1976).

In this case, equation (4) changes to:

$$\delta_t = \delta + \phi\left(s_t - s_0\right) \quad (9)$$
$$= \widetilde{\delta} + \phi s_t.$$

With this parameterization, the model is mean reverting in levels. This model has been widely used in the literature (see Bjerksund and Ekern (1995), and Schwartz (1997) model 1). Thus, geometric Brownian motion and mean reversion in levels are special cases of the model presented in this paper.

By construction, the convenience yield is instantaneously perfectly correlated to spot returns. As pointed out above, this seems a reasonable approximation for certain commodities, like oil and copper. On the other hand, the model can be seen as a means of obtaining simultaneously non-stationarity and mean reversion while keeping the market complete. To see how nonstationarity and mean reversion play together in the model, let's integrate equation (5) to find the spot log return between any to dates t and $t+\tau$:

$$s_{t+\tau} - s_t = \frac{\omega}{\omega+\phi}\left(\mu - \delta - \frac{1}{2}\sigma^2\right)\tau - \frac{\phi}{\omega+\phi}\left(m_t - \theta\right)\left(1 - e^{-(\omega+\phi)\tau}\right) + \quad (10)$$
$$\sigma \int_t^{t+\tau} \left[1 - \frac{\phi}{\omega+\phi}\left(1 - e^{-(\omega+\phi)(t+\tau-u)}\right)\right] dW_u.$$

The expression in the integral inside the brackets gives the "term structure of shocks". A shock that occurred at t has a residual impact on $s_{t+\tau}$ of $1 - \frac{\phi}{\omega+\phi}\left(1 - e^{-(\omega+\phi)\tau}\right)$. As τ grows without bound, this residual impact converges to:

$$1 - \frac{\phi}{\omega+\phi} = \frac{\omega}{\omega+\phi}. \quad (11)$$

There are three cases to consider: 1) $\phi = 0$, 2) $\phi > 0$ and $\omega > 0$, 3) $\phi > 0$ and $\omega = 0$[11]. When $\phi = 0$, shocks have permanent effects, and their residual impact is exactly 1. In this case, the spot is a random walk. When both ϕ and ω are positive, the residual impact of a shock experienced at t, as τ grows without bound, is $\frac{\omega}{\omega+\phi} < 1$. Shocks still have permanent effects, as with the random walk, but they partially vanish in the long run. This is the case in which the process generates simultaneously nonstationarity and mean reversion. Finally, when $\phi > 0$ and $\omega = 0$, the effect of shocks completely vanish in the long run, and the process is mean reverting in levels.

[11]The case $\phi < 0$ implies return continuation or momentum and will not be analyzed in this paper. For an analysis of this case applied to a stock index, see Rodriguez (2007).

The financial market is naturally complete through the dependence of m_t on W_t, the spot source of risk. Assume additionally that there are no arbitrage opportunities. Then, there exists a unique probability measure Q, equivalent to Π, such that the discounted prices of the spot (cum dividend) and of other traded assets are martingales under Q (Harrison and Kreps (1979)). In the next section I obtain the spot price process under the Q-measure, and derive formulas for futures prices. It turns out that these formulas are preference-free, an important feature of the model that allows to price futures and European options without the need to estimate the risk premium.

3 The price process under the Q-measure

In this section I obtain the risk-neutral spot price process and derive a closed form solutions for futures prices. I show that, under the Q-measure, the spot price does not depend on the risk premium. This implies that futures and European options[12] formulas are preference-free. Market completeness is not the only source of this result. Lioui (2006) has studied the general problem of pricing and hedging derivative securities when the underlying asset pays a stochastic dividend yield, and concluded that a risk premium has to be specified, even when the spot and the dividend yield are driven by the same source of risk. As it is shown below, what drives the result in the model studied in this paper is the special structure of equation (4), in which the convenience yield is characterized as an affine function of spot past performance.

Equation (1) defines μ as the total expected return on the commodity (capital gains plus convenience yield). As in Schwartz (1997), μ is assumed constant. Define now r as the constant instantaneous risk-free interest rate, and λ as the risk premium. Then, the total expected return can be decomposed as:

$$\mu \equiv r + \lambda \tag{12}$$

Plugging (12) back in (1) gives the risk-neutralized commodity price process:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r - \delta)\, dt - \phi m_t dt + \sigma \left(\frac{\lambda}{\sigma} dt + dW_t \right) \\ &= (r - \delta)\, dt - \phi m_t dt + \sigma dB_t, \end{aligned} \tag{13}$$

[12] European options are discussed in the next section.

where $B_t = \frac{\lambda}{\sigma}t + W_t$ is a Brownian motion under the Q-measure. Thus, the total expected return on the commodity under Q is r. More importantly, the risk-neutralized process for m_t does not depend on the risk premium either. To see this, plug (12) in (7) to get:

$$\begin{aligned} dm_t &= -(\omega+\phi)(m_t-\theta^*)\,dt + \sigma\left(\frac{\lambda}{\sigma}dt + dW_t\right) \\ &= -(\omega+\phi)(m_t-\theta^*)\,dt + \sigma dB_t, \end{aligned} \tag{14}$$

where now:

$$\theta^* = \frac{r - \frac{1}{2}\sigma^2 - \delta}{\omega+\phi}. \tag{15}$$

So neither S_t nor m_t depend on λ under Q. As a consequence, the model renders preference-free formulas for contingent claims.

To solve for the spot price under Q replace μ with r in equation (10). This shows that the commodity price is a lognormal process under the risk-neutral measure[13]. That is:

$$\ln(S_T)\,\tilde{}\,N(s_t+\Omega_\tau, \Sigma_\tau), \tag{16}$$

where T is maturity time. From equation (10) we have that (under Q):

$$\Omega_\tau = \frac{\omega}{\omega+\phi}\left(r-\delta-\frac{1}{2}\sigma^2\right)\tau - \frac{\phi}{\omega+\phi}(m_t-\theta^*)\left(1-e^{-(\omega+\phi)\tau}\right), \tag{17}$$

and:

$$\Sigma_\tau = \frac{\sigma^2}{(\omega+\phi)^2}\left(\omega^2\tau + \frac{2\phi\omega}{\omega+\phi}\left(1-e^{-(\omega+\phi)\tau}\right) + \frac{\phi^2}{2(\omega+\phi)}\left(1-e^{-2(\omega+\phi)\tau}\right)\right), \tag{18}$$

where $\tau = T - t$ is the time before maturity. Note that if $\phi = 0$, $\Sigma_\tau = \sigma^2\tau$. That is, the variance grows linearly with time to maturity, which corresponds to the random walk case. On the other hand, if $\phi > 0$ and $\omega = 0$, $\Sigma_\tau = \frac{\sigma^2}{2\phi}\left(1-e^{-2\phi\tau}\right)$, the variance of a mean reverting process with reversion rate ϕ.

The forward price[14] for delivery of one unit of the commodity τ periods ahead is the

[13] As both μ and r are constant, the commodity price is a lognormal process under the statistical measure as well.

[14] Note that the words "futures price" and "forward price" can be used interchangeably in this context,

expected commodity price under the risk-neutral measure. Given the normality of log (S_t) under Q, the forward price is easily obtained in closed form:

$$\begin{aligned} F_\tau &= E_t^Q\left(S_T\right) \\ &= S_t \times \exp\left(\Omega_\tau + \frac{1}{2}\Sigma_\tau\right). \end{aligned} \tag{19}$$

From equations (17) and (18), this formula does not include the risk premium, and so it is preference-free.

The dynamics of the forward is described, after applying Ito's lemma, by the following differential equation:

$$\frac{dF_\tau}{F_\tau} = \sigma\left[1 - \frac{\phi}{\omega+\phi}\left(1 - e^{-(\omega+\phi)\tau}\right)\right] dB_t, \tag{20}$$

The forward price process has no drift under Q, because no money is paid to enter the contract. When $\phi = 0$, the volatility of the forward return is independent of time to maturity. This is the case in which the spot is a random walk. When $\phi > 0$, the volatility of futures returns decreases with the time to maturity, and it only dies down, as $\tau \to \infty$, when $\omega = 0$ (the mean reversion case). Otherwise, it converges to:

$$\sigma\left(1 - \frac{\phi}{\omega+\phi}\right) = \frac{\sigma\omega}{\omega+\phi} > 0. \tag{21}$$

The term structure of futures return volatilities is the same as the term structure of shocks (see equation (10)). This is consistent with Bessembinder et al. (1996) explanation of the Samuelson hypothesis.

4 Pricing options

The price of a European call option written on the spot, with maturity T and strike K, is the expectation under Q of its payoff at maturity, discounted by the risk-free rate:

$$C_t = e^{-r\tau} E_t^Q\left[Max\ \left(S_T - K, 0\right)\right]. \tag{22}$$

because they are equal under the current assumption of a constant risk-free rate.

Equation (22) can be written as:

$$C_t = e^{-r\tau} E_t^Q \left[(S_T\ ) \times \mathbf{1}_{\{S_T > K\}} \right] - e^{-r\tau} K P^Q \left(S_T > K \right), \tag{23}$$

where $\mathbf{1}_{\{S_T > K\}}$ is the indicator function of the event $\{S_T > K\}$, $E_t^Q \left[(S_T\ ) \times \mathbf{1}_{\{S_T > K\}} \right]$ is the Q-expected value of the spot at maturity, conditioned on the event that the option will be exercised at maturity, and $P^Q(S_T > K)$ is the probability under Q of this event. Due to the normality of $\ln(S_t)$, the expectation in the first term of (23) can be solved as:

$$E_t^Q \left[(S_T\ ) \times \mathbf{1}_{\{S_T > K\}} \right] = S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1), \tag{24}$$

where $N(d_1)$ is the value of the Normal cummulative distribution fuction at d_1, and:

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \Omega_\tau + \Sigma_\tau}{\sqrt{\Sigma_\tau}}. \tag{25}$$

The probability of the option finishing in the money is:

$$P^Q(S_T > K) = N(d_2), \tag{26}$$

where:

$$d_2 = d_1 - \sqrt{\Sigma_\tau}. \tag{27}$$

So:

$$C_t = \left[S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1) - K N(d_2) \right] e^{-r\tau}. \tag{28}$$

It is important to note that this formula, as the formula for the forward price (19), does not include preference parameters.

The price of a European put on the same commodity can be found using put-call parity. That is, because buying a call and shorting a put, both with maturity T and strike K, is equivalent to having a long position in a forward contract with maturity T and forward price K, we can express the put price as:

$$P_t = C_t - \left[E_t^Q(S_T) - K \right] e^{-r\tau} \tag{29}$$

Plugging (19) and (28) in (29) we get:

$$P_t = \left[KN(-d_2) - S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(-d_1) \right] e^{-r\tau}. \tag{30}$$

4.1 The riskless hedge

The financial market in this paper is complete, so it is possible to construct a riskless hedge by continuously trading in the commodity and a riskless bond. This section shows how to construct such riskless hedge. Although the assumption of continuous trading may be too strong for commodities, I include here the riskless hedge for completeness, and also as a means to derive the fundamental partial differential equation that contingent claims must satisfy in the model[15].

Assume that a call has been written on the commodity and that a hedging portfolio is started consisting on the shorted call and a long position in the underlying commodity. The initial value of the portfolio is:

$$\Pi_t = \Delta S_t - C(S_t, m_t, t). \tag{31}$$

where Δ is the number of long units of the commodity. The change in the value of the portfolio over the next period is:

$$\begin{aligned} d\Pi_t &= \Delta dS_t + \Delta(\delta + \phi m_t) S_t dt \\ &- \frac{\partial C}{\partial S} dS_t - \frac{\partial C}{\partial m}\frac{1}{S_t}\left[dS_t - S_t\left(\frac{1}{2}\sigma^2 + \omega m_t\right) dt \right] \\ &- \frac{\partial C}{\partial t} dt - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 dt - \frac{1}{2}\frac{\partial^2 C}{\partial m^2}\sigma^2 dt - \frac{\partial^2 C}{\partial S \partial m}\sigma^2 dt, \end{aligned} \tag{32}$$

where in the fourth term of the equation I use the fact that

$$ds_t = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt. \tag{33}$$

The risk in the portfolio comes from its exposure to W_t. To eliminate this risk, choose:

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial m}\frac{1}{S_t}. \tag{34}$$

[15] The formula could also be derived using an equilibrium argument, following Cox, Ingersoll and Ross (1985). See, for example Gibson and Schwartz (1990) as an example of this procedure.

Note that to hedge the position it is necessary to eliminate not only the risk coming from random changes in the spot (the first term in (34)), but also the risk coming from the stochastic dividend yield (the second term in (34)). So plugging (34) in (32) cancels the portfolio's overall exposure to W_t. As the portfolio is now riskless, it must earn the riskless interest rate to preclude arbitrage:

$$\left(\frac{\partial C}{\partial S} + \frac{\partial C}{\partial m}\frac{1}{S_t}\right)(\delta + \phi m_t) S_t + \frac{\partial C}{\partial m}\omega m_t - A(t) = r\Pi_t, \tag{35}$$

where:

$$A(t) = \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 + \frac{1}{2}\frac{\partial^2 C}{\partial m^2}\sigma^2 dt + \frac{\partial^2 C}{\partial S \partial m}\sigma^2. \tag{36}$$

Operating on (35) we get:

$$\frac{\partial C}{\partial S} S_t (r - \delta - \phi m_t) + \frac{\partial C}{\partial m}\left(-(\omega + \phi) m_t + r - \delta - \frac{1}{2}\sigma^2\right) + A(t) - rC = 0, \tag{37}$$

where the term in parenthesis multiplying $\frac{\partial C}{\partial m}$ can be written as: $-(\omega + \phi)(m_t - \theta^*)$. So (37) is the fundamental partial differential equation that all contingent claims written on the spot must satisfy.

It is possible to calculate the Δ of the call in closed form using equations (28) and (34). We have:

$$\frac{\partial C}{\partial S} = e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1) e^{-r\tau}, \tag{38}$$

and:

$$\frac{\partial C}{\partial m} = -S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N(d_1) \frac{\phi}{\omega + \phi}\left(1 - e^{-(\omega+\phi)\tau}\right) e^{-r\tau}. \tag{39}$$

Therefore:

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial m}\frac{1}{S_t} = \frac{\partial C}{\partial S}\left[1 - \frac{\phi}{\omega + \phi}\left(1 - e^{-(\omega+\phi)\tau}\right)\right]. \tag{40}$$

Note that $\Delta \geq 0$. Also, as expected, $\Delta \to \left\{\begin{smallmatrix} 1 \text{ if } S_t > K \\ 0 \text{ if } S_t \leq K \end{smallmatrix}\right.$ as $\tau \to 0$.

Similarly, it is possible to calculate the other "Greeks". In particular, the Gamma of the option can be expressed as:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}\left[1 - \frac{\phi}{\omega + \phi}\left(1 - e^{-(\omega+\phi)\tau}\right)\right], \tag{41}$$

where:

$$\frac{\partial^2 C}{\partial S^2} = e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N'(d_1) \frac{\partial d_1}{\partial S} e^{-r\tau} \tag{42}$$
$$= e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} \frac{N'(d_1)}{S\sqrt{\Sigma_\tau}} e^{-r\tau}$$

Finally, the Vega of the option is:

$$\nu = \frac{\partial C}{\partial \sigma} = \left[\frac{\partial S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau}}{\partial \sigma} N(d_1) + S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau} N'(d_1) \frac{\Sigma_\tau}{\sigma} \right] e^{-r\tau}. \tag{43}$$

For details about these formulas and their derivation, see the Appendix.

4.2 Pricing options on futures

In the case of commodities, it is usually easier to observe futures rather than spot prices. It is even the case that in some exchanges the nearest maturity futures price is taken as a proxy for the spot price. So, for many commodities options are not written directly on the spot, but on the futures price, and, as a consequence, it is not uncommon in the literature to deal directly with the pricing of options on futures. This section shows how to use the model introduced in section 2 to price this kind of options.

If the maturity of the option and the maturity of the futures contract are the same, the current futures price $F_\tau = F(t, T)$ can be used to price options on the spot. So, as $F(T, T) = S_T$, the call price in equation (28) can be rewritten as:

$$C(t, T) = [F_\tau N(d_1) - KN(d_2)] e^{-r\tau}, \tag{44}$$

with:

$$d_1 = \frac{\log\left(\frac{F_\tau}{K}\right) + \frac{1}{2}\Sigma_\tau}{\sqrt{\Sigma_\tau}}, \tag{45}$$

where Σ_τ is as defined in equation (18). On the other hand, suppose that T is the maturity time of the futures contract, and that the option matures at $s < T$. Then, as integration of equation (13) must be done over the life of the option, the variance in equation (18) has to be replaced by:

$$\Sigma_{\tau}^{*}=\frac{\sigma^{2}}{(\omega+\phi)^{2}}\left(\begin{array}{c}\omega^{2}(s-t)+\frac{2\phi\omega}{\omega+\phi}\left(e^{-(\omega+\phi)(T-s)}-e^{-(\omega+\phi)(T-t)}\right)+\\ \frac{\phi^{2}}{2(\omega+\phi)}\left(e^{-2(\omega+\phi)(T-s)}-e^{-2(\omega+\phi)(T-t)}\right)\end{array}\right) \tag{46}$$

In pricing derivatives it is important that the model being used can be calibrated to the current futures curve. Then, as equation (45) shows, pricing accurately is a matter of accurate variance estimation. By making δ a deterministic function of time, the model studied in this paper can be calibrated to the current futures curve. Let $f_{\tau}=\ln F(t,T)$. When δ is a deterministic function of time, we have, after taking logs on (19):

$$f_{\tau}-f_{t}=\int_{t}^{T}\gamma(u)\,du+b(T-t), \tag{47}$$

where:

$$\int_{t}^{T}\gamma(u)\,du=-\int_{t}^{T}\delta(u)\,du+\phi\int_{t}^{T}\int_{t}^{s}\delta(u)\,e^{-(\omega+\phi)(s-u)}duds, \tag{48}$$

and:

$$\begin{aligned}b(T-t)=&\frac{\omega}{\omega+\phi}\left(r-\frac{1}{2}\sigma^{2}-\widetilde{\theta}\right)(T-t)-\frac{\phi}{\omega+\phi}\left(m_{t}-\widetilde{\theta}\right)\left(1-e^{-(\omega+\phi)(T-u)}\right)+\\ &\sigma\int_{t}^{T}\left[1-\frac{\phi}{\omega+\phi}\left(1-e^{-(\omega+\phi)(T-u)}\right)\right]dW_{u},\end{aligned} \tag{49}$$

with $\widetilde{\theta}=\frac{r-\frac{1}{2}\sigma^{2}}{\omega+\phi}$ in this case.

Now define F_{τ}^{M} as the futures price at t observed in the market for delivery at $T=t+\tau$, and F_{τ} the corresponding futures price generated by the model. Showing that the model can be fitted to any given futures price curve is equivalent to show that it is possible to choose the function $\gamma(T)$ such that:

$$F_{\tau}=F_{\tau}^{M} \tag{50}$$

for all T. Taking logs on both sides, replacing equation (46), and solving for $\int_{t}^{t+\tau}\gamma(u)\,du$ we get:

$$\int_t^T \gamma(u)\, du = f_\tau^M - b(T-t) - f_t.$$

and then:

$$\gamma(T) = \frac{\partial f_\tau^M}{\partial T} - \frac{\partial b(T-t)}{\partial T}. \tag{51}$$

That is, the model can in principle be made consistent with any futures price curve.

Equation (45) assumes that the parameters σ, ω, and ϕ are known, or have already been estimated. The next section studies how to estimate the parameters of the model.

5 Empirical Results

There are two approaches to the pricing of commodity derivatives. The first starts with a fully specification of the process followed by the spot price and by other state variables such as the convenience yield, and then derives the prices of contingent claims that are consistent with those fundamental processes. An example of this approach is the paper by Schwartz (1997). The second approach takes the entire futures curve as a primitive, and obtains derivative prices that are consistent with it. Examples of this approach are Amin, Ng and Pirron (1995) and Miltersen and Schwartz (1998).

In the first approach, the goal is to recover, as accurately as possible, the futures prices observed in the market, in the understanding that a model that accurately reproduces what is observed can be trusted to predict what is not observed, such as prices for long term maturities useful in real options analysis. Implementation in this case requires the estimation of model parameters (including risk premiums) using futures price data. The problem is that the state variables of the model -the spot price and the convenience yield- are usually unobservable. Schwartz (1997) shows that the Kalman filter can be used for parameter estimation and the recovery of state variables from futures price data, and this technique is still widely used in the literature.

In the second approach, the parameters are calibrated to make the model fit a given set of observed derivative prices. The parameters are then used to value other derivatives whose prices are not observed. The interest is in the calibration of spot volatility parameters; there is no need to calibrate parameters affecting only the level of the derivative price, because this price is recovered from market data[16]. The fact that in this case prices

[16] This is similar to the no arbitrage technique pioneered in the term structure literature by Ho and Lee (1986).

are considered as given is what differentiates the two approaches.

This section provides an empirical assessment of the model. To avoid repetition, I refer from now on to the model introduced in this paper as the "m-model", and to its restricted version, mean reversion in levels, as the "mr-model". First I implement the Kalman filter to obtain estimates of the parameters, and investigate performance by means of pricing errors. Next, I use the function $\delta(t)$ to fit the model to the current term structure of futures prices, and calibrate the remaining parameters to the term structure of futures return volatility.

The remainder of the section is organized as follows. Section 5.1 describes the data. In section 5.2 the model parameters are estimated using the Kalman filter. Finally, section 5.3 presents the calibration of the model to the term structure of futures return volatilities.

5.1 Data

The models are implemented on a data set consisting of weekly observations of futures prices of oil (NYMEX WTI). Daily data was originally obtained from Bloomberg and then transformed into weekly by choosing every Wednesday observation. Eleven maturities were used in the empirical exercises, going from the contract closest to maturity (F1) to a longer term contract (F11). The shortest maturity is about two weeks; the longest maturity, less than two years. For each contract there are 249 observations, starting on March 17, 1999, and ending on December 31, 2003. The interest rate is assumed constant and fixed at 4%. The data is described in Table 1:

Table 1: Oil Data Description

Futures Contract	Mean Price (Standard Error)	Mean Maturity (Standard Error)	Standard Dev. of futures return
F1	27.05 (4.72)	0.043 (0.024)	0.373
F2	26.41 (4.19)	0.210 (0.024)	0.313
F3	25.64 (3.81)	0.377 (0.024)	0.265
F4	25.03 (3.57)	0.544 (0.024)	0.235
F5	24.44 (3.37)	0.711 (0.024)	0.216
F6	23.94 (3.19)	0.878 (0.024)	0.199
F7	23.54 (3.06)	1.045 (0.024)	0.186
F8	23.16 (2.92)	1.212 (0.024)	0.175
F9	22.84 (2.82)	1.379 (0.024)	0.169
F10	22.58 (2.72)	1.546 (0.024)	0.161
F11	22.39 (2.66)	1.713 (0.024)	0.159

The mean prices go down uniformly with maturity. The futures returns are calculated as the difference between the log of the futures prices, and their volatilities also decrease steadily with maturity.

5.2 Parameter estimation

To estimate the model's parameters by means of the Kalman filter it is necessary first to express the model in state-space form. The measurement equation is:

$$y_t = d_t + Z_t \times \begin{bmatrix} S_t \\ m_t \end{bmatrix} + \varepsilon_t, \qquad t = 1, ..., NT$$

where T is the number of observations, $N = 11$ is the number of maturities, and:

$$y_t = \ln\left(F_{\tau_i}\right) \qquad i = 1, ..., N$$

is 11×1 vector of observable log futures prices. Also, d_t and Z_t are 11×1 and 11×2 matrices:

$$d_t = \begin{bmatrix} \left(\frac{\omega}{\omega+\phi}\left(r - \delta - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\frac{\sigma^2}{(\omega+\phi)^2}\omega^2\right)\tau_i + \frac{\phi}{\omega+\phi}\left(\theta^* + \omega\frac{\sigma^2}{(\omega+\phi)^2}\right)\left(1 - e^{-(\omega+\phi)\tau_i}\right) + ... \\ ... + \frac{1}{4}\frac{\sigma^2}{(\omega+\phi)^2}\frac{\phi^2}{\omega+\phi}\left(1 - e^{-2(\omega+\phi)\tau_i}\right) \end{bmatrix}$$

and:

$$Z_t = \left[1, \frac{\phi}{\omega + \phi}\left(1 - e^{-(\omega+\phi)\tau_i}\right)\right], \qquad i = 1, ..., N$$

The observations error, ε_t, is normally distributed with zero mean. To reduce the number of parameters to estimate, the variance of the observation error is assumed constant:

$$Var\left(\varepsilon_t\right) = \sigma_\varepsilon^2 \mathbf{I}_N,$$

where $\mathbf{I}_N$ is the N-dimensional identity matrix.

The assumption of a constant variance of the observation error is common in studies using a large number of maturities (see for example Sorensen (2002) and Cortazar and Naranjo (2006)). The corresponding estimate gives the average variance across maturities. For the sample studied in this paper, the assumption leads to a slight improvement in the fit at the short and long ends of the futures curves, at the expense of the fit at the intermediate maturities, but otherwise it does not have a significant effect on the estimates of the parameters.

The transition equations describe the dynamics of the discretized state variables:

$$[S_t, m_t]^T = c_t + Q_t \times \begin{bmatrix} S_{t-1} \\ m_{t-1} \end{bmatrix} + \eta_t,$$

where:

$$c_t = \left[\left(\mu - \frac{1}{2}\sigma^2 - \delta\right)\Delta t, \left(\omega + \phi\right)\theta\Delta t\right],$$

$$Q_t = \begin{bmatrix} 1 & -\phi\Delta t \\ 0 & 1 - \left(\omega + \phi\right)\Delta t \end{bmatrix},$$

and η_t is normally distributed with:

$$E\left(\eta_t\right) = 0, \quad Var\left(\eta_t\right) = \sigma^2 \mathbf{O}_2,$$

where $\mathbf{O}_2$ is a 2×2 matrix of ones.

Estimation results are presented in Table 2. Estimates for the m-model are shown in the second column; estimates for the mr-model are shown in the fourth column. The mr-model is obtained from the m-model by imposing $\omega = 0$.

Table 2: Estimation Results

Parameters	M-Model	Std.Dev.	MR-Model	Std.Dev.
μ	0.5018	(0.1673)	0.3680	(0.1018)
δ	0.1421	(0.0040)	0.0306	(0.0776)
σ	0.3653	(0.0203)	0.2226	(0.0110)
ϕ	0.9780	(0.0237)	0.2410	(0.0089)
ω	0.6323	(0.0172)	0.00	
σ_ε	0.0222	(0.0003)	0.0335	(0.0005)
Likelihood	6155.2		5139.2	

For the m-model, the value of the likelihood function is 6155.2. The parameter ω, which measures the weight of past spot returns in the convenience yield, is positive and significant. As discussed in section 2, a positive ω means that shocks are only partially reversed in the long run, suggesting that mean reversion in levels is not an adequate model for the data. The loading of m_t, ϕ, and σ, the instantaneous volatility of spot returns, are also positive and significant. These three parameters imply that the volatility of futures returns converges in the long run to:

$$\frac{\sigma \times \omega}{\phi + \omega} = 0.1434.$$

The total return on the spot, μ, is 0.5018, while δ is equal to 0.1252. Both parameters are significant. The implied long run convenience yield is:

$$\delta \times \frac{\omega}{\phi + \omega} + \left(\mu - \frac{1}{2}\sigma^2\right) \times \frac{\phi}{\phi + \omega} = 0.3982.$$

Finally, σ_ε, the measurement error, is also positive and significant[17].

Imposing $\omega = 0$ reduces the value of the likelihood function from 6155.2 to 5139.2. A likelihood ratio test shows that this difference is strongly significant, with negligible p-value. Only two out of the three parameters of the mean reversion in levels model, ϕ and σ, are significant.

It is apparent from Table 2 that imposing $\omega = 0$ worsens the model's ability to fit the data. Table 3 explores how this result translates into the pricing errors generated by the models. Following standard practice in the literature, pricing errors are measured by

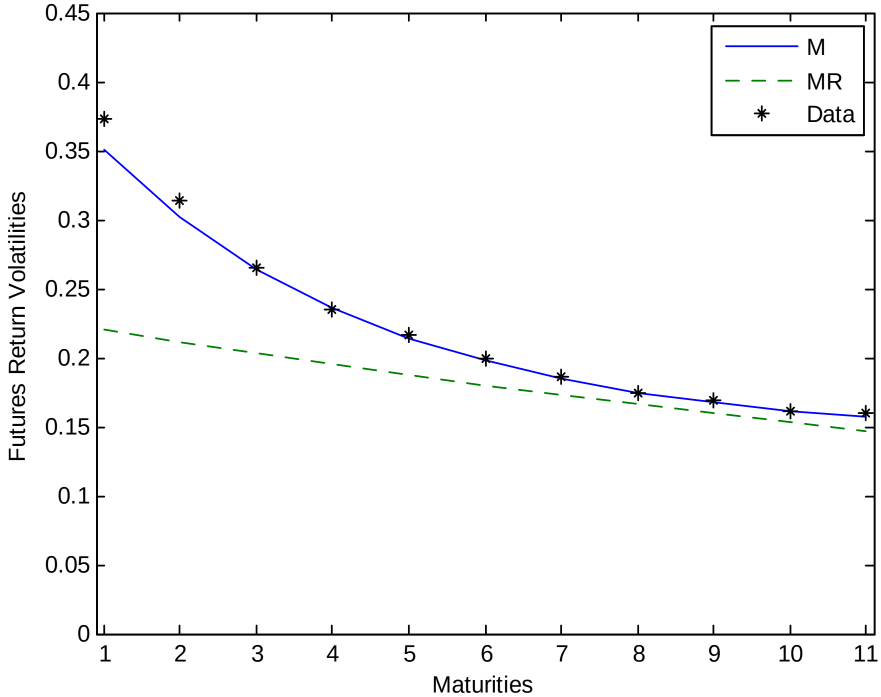
[17] Hodges and Ribeiro (2004) estimate Schwartz (1997) model 2 over a similar period and obtain estimates that are consistent with mine. They get $\mu = 0.50$, and the long run convenience yield equal to 0.356.

the average mean square error (RMSE) and the absolute mean error (AME). The mean reverting model is a restriction of the m-model, and so it generates larger pricing errors. For both models, pricing errors are larger at the short and long ends of the price curve. However, measured by the RMSE, the average dollar pricing error of the mr-model is almost 46% larger than the m-model, while measured by the AME they are 53% larger.

Table 3: Cross-Section comparison between models

	RMSE		AME	
Model Contract	MR	M	MR	M
Panel A: In Dollars				
F1	1.570	1.151	1.245	0.867
F2	0.981	0.650	0.813	0.462
F3	0.558	0.305	0.481	0.245
F4	0.335	0.269	0.286	0.217
F5	0.291	0.309	0.233	0.230
F6	0.346	0.362	0.272	0.267
F7	0.420	0.399	0.336	0.298
F8	0.540	0.437	0.446	0.333
F9	0.673	0.489	0.574	0.378
F10	0.826	0.536	0.713	0.427
F11	1.015	0.574	0.881	0.469
All	0.687	0.498	0.571	0.381
Panel B: In Percentage				
F1	5.829	3.851	4.614	3.058
F2	3.827	2.308	3.132	1.697
F3	2.234	1.189	1.888	0.964
F4	1.327	1.060	1.131	0.873
F5	1.140	1.207	0.928	0.933
F6	1.450	1.445	1.141	1.097
F7	1.851	1.629	1.462	1.247
F8	2.432	1.828	1.972	1.424
F9	3.069	2.094	2.567	1.655
F10	3.807	2.373	3.239	1.919
F11	4.708	2.630	4.051	2.155
All	2.879	1.965	2.375	1.548

Figure 1: Futures Return Volatilties



The models are also compared on how well they match the term structure of futures return volatilities. The results are shown in figure 1. The mr-model misses completely the curve of empirical returns volatilities, a result that is also obtained in Schwartz (1997) for a different sample. The m-model, instead, matches the empirical term structure quite well, although it understates the first two shortest maturities. The improvement in the fit is remarkable, especially considering that the m-model has only one additional parameter.

5.2.1 Model calibration

When the goal is to price illiquid options, the models can be implemented by first fitting the current forward curve, and then by estimating the remaining parameters using futures or liquid options data. Section 4.2 showed that, if δ is made a deterministic function of time, the m-model can be adjusted to fit any forward curve. In this section I estimate the remaining parameters, ω, ϕ and σ, by forcing the model to match the term structure of futures return volatilities. I present results for both the m and the mr-models, which is obtained by retrincting the m-model to $\omega = 0$.

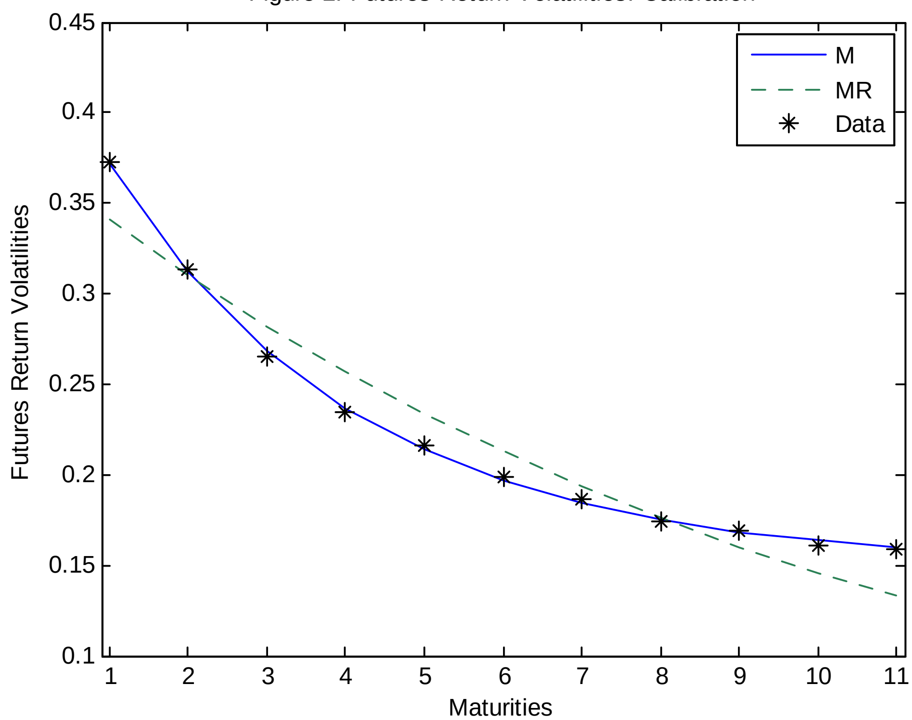
Table 4 shows the results. The estimates are different from the ones obtained in the previous section, because now the parameters no longer have to determine the futures price levels. Given the values of the other parameters, this task is undertaken by the function $\delta(t)$ alone. Therefore, the parameters can be chosen in such a way that the fit of volatilities is greatly improved. Estimates are obtained by minimizing the square of the difference between the empirical and theoretical futures return volatilities over all maturities.

Table 4: Parameter Calibration

Parameter	M-Model	MR-Model
σ	0.3904	0.3489
ϕ	1.1529	0.5641
ω	0.7219	0.00

Figure 2 shows that the m-model is able to fit the term structure perfectly, while the mean reverting model overestimates the mid-term volatilities, and underestimates the volatilities of the shortest and longest maturities.

Figure 2: Futures Return Volatilities. Calibration



6 Conclusions

This paper presents a complete market model of commodity prices that exhibits price nonstationarity and mean reversion under the martingale measure, and, as a consequence, it is able to fit a slowly decaying term structure of futures return volatilities. The model has mean reversion in levels and geometric Brownian motion as special cases, and renders preference-free formulas for the prices of futures contracts and European options.

Implemented on a sample of oil futures prices, the model generates substantially lower pricing errors than the mean reversion in levels model, and is capable to produce a perfect fit of the term structure of futures return volatilities.

The model is parsimonous (it has just one more parameter than the mean reversion in levels model) and provides a useful benchmark to value complex contracts for which no closed form solutions are known. On this regard, it can be seen as a good alternative to widely used one-factor models.

References

[1] Amin, K., V. Ng, and S.C. Pirrong, 1995, "Valuing Energy Derivatives," chapter 3 in Managing Energy Price Risk, 57-70, London: Risk Publications.

[2] Bessembinder H., 1993, An empirical analysis of risk premia in futures markets. Journal of Futures Markets, 13, 6, 611-630.

[3] Bessembinder, H., Coughenour, J.F., Seguin, P.J., Smoller, M.M., 1995, Mean reversion inequilibrium asset prices: evidence from the futures term structure. Journal of Finance 50, 361-375.

[4] Bessembinder, H., Coughenour, J.F., Seguin, P.J., Smoller, M.M., 1996, Is there a term structure offutures volatilities? reevaluating the Samuelson hypothesis. Journal of Derivatives, Winter, 45-58.

[5] Bjerksund, P., Ekern, S., 1995, Contingent Claims Evaluation of Mean-Reverting Cash Flows in Shipping, in Trigeorgis, Lenos (Ed.): Real Options in Capital Investment, Praeger, London, pp. 207-219.

[6] Black, F., 1976, The Pricing of Commodity Contracts. Journal of Finance Economics, 167-179.

[7] Brennan M.J., 1958, The supply of storage. American Economic Review, 47, 50-72.

[8] Brennan M.J., & Schwartz E.S, 1985, Evaluating natural resource investments. Journal of Business, 58, 135-157.

[9] Brennan M.J.,1991, The price of convenience and the valuation of commodity contingent claims, in D. Land, B. Oksendal (Ed.), Stochastic Models and Options Values, Elsevier Science Publishers.

[10] Cortazar, G., Naranjo, L.,2006, An N-Factor Gaussian Model of Oil Futures Prices. The Journal of Futures Markets, Vol.26, No. 3, March, 243-268.

[11] Cox, J. C., Ingersoll, J. E. & Ross, S. A., 1985, An intertemporal general equilibrium model of asset prices, Econometrica 53, 363-384.

[12] Gibson R. & Schwartz E.S., 1990, Stochastic convenience yield and the pricing of oil contingent claims. Journal of Finance, 45, 959-975.

[13] Harrison, M. & Kreps, D., 1979, Martingales and arbitrage in multiperiod security markets, J. Economic Theory 20, 381–408.

[14] Hilliard, J.E., & Reis, J., 1998, Valuation of commodity futures and options under stochastic convenience yield, interest rates, and jump diffusions in the spot. Journal of Financial and Quantitative Analysis, 33, 61-86.

[15] Ho, T., Lee, S., 1986, Term Structure Movements and Pricing Interest Rate Contingent Claims, Journal of Finance, 41, 1011-1029.

[16] Hodges, S., Ribeiro, D., 2004, A Two-Factor Model for Commodity Prices and Futures Valuation. Unpublished manuscript. Warwick Business School.

[17] Lioui, A., 2006, Black-Scholes-Merton revisited under stochastic dividend yields. Journal of Futures Markets, Vol 26, Issue 7, 703-732.

[18] Miltersen K.R. & Schwartz E.S., 1998, Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates. Journal of Financial and Quantitative Analysis, 33, 1, 33-59, March.

[19] Richter M.C. & Sorensen C., 2002, Stochastic volatility and seasonality in commodity futures and options: the case of soybeans. Journal of Futures Markets.

[20] Rodríguez, J.C., 2007, Option Pricing and Momentum. Unpublished manuscript. Tilburg University.

[21] Routledge, B., D. J. Seppi, and C. S. Spatt, 1998, Equilibrium Forward Curves for Commodities, Working Paper, Carnegie Mellon University.

[22] Schwartz E.S., 1997, The stochastic behavior of commodity prices: implications for valuation and hedging. Journal of Finance, 52, 923-973.

[23] Schwartz E.S., 1998, Valuing long-term commodity assets. Journal of Energy Finance and Development, 3, 85-99.

[24] Sorensen C., 2002, Modeling seasonality in agricultural commodity futures. Journal of Futures Markets 22(5): 393–426.

[25] Yan, X., 2002, Valuation of commodity derivatives in a new multi-factor model. Review of Derivatives Research, Volume 5, Number 3, October.

7 Appendix. Derivation of delta, gamma, and vega

The following lemma will be useful in the derivation of delta and vega:

Lemma 1 *Define* $F_\tau = S_t e^{\Omega_\tau + \frac{1}{2}\Sigma_\tau}$*. Then:*

$$F_\tau N'(d_1) - KN'(d_2) = 0,$$

where d_1 *and* d_2 *are as in equations (25) and (27).*

Proof. *Recall that:*

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and write $d_1 = d_2 + \sqrt{\Sigma_\tau}$*. Then:*

$$\begin{aligned} N'(d_1) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_2^2 + 2d_2\sqrt{\Sigma_\tau} + \Sigma_\tau}{2}\right) \\ &= N'(d_2) \exp\left(-d_2\sqrt{\Sigma_\tau} - \frac{1}{2}\Sigma_\tau\right) \\ &= N'(d_2) \frac{K}{F_\tau}. \end{aligned}$$

So:

$$F_\tau N'(d_1) - KN'(d_2) = F_\tau N'(d_2) \frac{K}{F_\tau} - KN'(d_2) = 0,$$

■

Now it is straightforward to derive the Greeks:

Lemma 2 ***Lemma 3***

$$\Delta = \frac{\partial C}{\partial S}\left[1 - \frac{\phi}{\omega + \phi}\left(1 - e^{-(\omega+\phi)\tau}\right)\right]$$

Proof. *Recall that:*

$$\Delta = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial m}\frac{1}{S_t}.$$

Fron equation (28) we have:

$$\frac{\partial C}{\partial S}=\left[\frac{\partial F_\tau}{\partial S}N\left(d_1\right)+F_\tau N'\left(d_1\right)\frac{\partial d_1}{\partial S}-KN'\left(d_2\right)\frac{\partial d_2}{\partial S}\right]e^{-r\tau}.$$

Noting that $\frac{\partial d_1}{\partial S}=\frac{\partial d_2}{\partial S}$,

$$\frac{\partial C}{\partial S}=\left[\frac{\partial F_\tau}{\partial S}N\left(d_1\right)+\left[F_\tau N'\left(d_1\right)-KN'\left(d_2\right)\right]\frac{\partial d_2}{\partial S}\right]e^{-r\tau},$$

which, from Lemma 1 is:

$$\frac{\partial C}{\partial S}=\frac{\partial F_\tau}{\partial S}N\left(d_1\right)e^{-r\tau}.$$

Now,

$$\frac{\partial C}{\partial m}=\left[\frac{\partial F_\tau}{\partial m}N\left(d_1\right)+F_\tau N'\left(d_1\right)\frac{\partial d_1}{\partial S}\frac{\partial S}{\partial m}-KN'\left(d_2\right)\frac{\partial d_2}{\partial S}\frac{\partial S}{\partial m}\right]e^{-r\tau}.$$

Proceeding as before, we get:

$$\frac{\partial C}{\partial m}=\frac{\partial F_\tau}{\partial m}N\left(d_1\right)e^{-r\tau}.$$

Noting that:

$$\frac{\partial F_\tau}{\partial m}=-S\frac{\partial C}{\partial S}\frac{\phi}{\omega+\phi}\left(1-e^{-(\omega+\phi)\tau}\right),$$

the result follows. ■

Gamma follows immediatly from this last result:

Lemma 4

$$\Gamma=\frac{\partial\Delta}{\partial S}=\frac{\partial^2 C}{\partial S^2}\left[1-\frac{\phi}{\omega+\phi}\left(1-e^{-(\omega+\phi)\tau}\right)\right]$$

Proof. *Ommited.* ■

Finally, the derivation of Vega:

Lemma 5

$$\nu = \left[\frac{\partial F_\tau}{\partial \sigma} N(d_1) + F_\tau N'(d_1) \frac{\Sigma_\tau}{\sigma} \right] e^{-r\tau},$$

where:

$$\frac{\partial F_\tau}{\partial \sigma} = \left(\frac{\partial \Omega_\tau}{\partial \sigma} + \frac{1}{2} \frac{\partial \Sigma_\tau}{\partial \sigma} \right) F_\tau,$$

and:

$$\frac{\partial \Omega_\tau}{\partial \sigma} = -\left(\frac{\omega}{\omega + \phi} + \frac{\phi}{\omega + \phi} \frac{1 - e^{-(\omega+\phi)\tau}}{\omega + \phi} \right) \sigma$$
$$\frac{\partial \Sigma_\tau}{\partial \sigma} = \frac{2\Sigma_\tau}{\sigma}$$

Proof. *First, take the derivative of the call with respect to* σ :

$$\frac{\partial C}{\partial \sigma} = \left[\frac{\partial F_\tau}{\partial \sigma} N(d_1) + F_\tau N'(d_1) \frac{\partial d_1}{\partial \sigma} - K N'(d_2) \frac{\partial d_2}{\partial \sigma} \right] e^{-r\tau}. \tag{52}$$

Then, note that:

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \frac{\partial \sqrt{\Sigma_\tau}}{\partial \sigma}. \tag{53}$$

Now, plugging (53) back in (52), and noting that:

$$\frac{\partial \sqrt{\Sigma_\tau}}{\partial \sigma} = \frac{\Sigma_\tau}{\sigma},$$

the lemma follows directly from Lemma 1. ■